Test module for flatness

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1. Test Modules for Injectivity and Projectivity

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   - Whitehead Test Modules

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Let M be a left R-module.

$$\mathcal{P}(M) = \{N \in R - Mod | \operatorname{Ext}^1_R(N, M) = 0\}$$

(1) $\mathcal{P}(M) = R - Mod$ iff M is injective.

(2) Let $\mathcal{P}rj$ be denote the class of all projective modules. $\mathcal{P}rj \subseteq \mathcal{P}(M)$ for each module M.

(3) Let Γ be a complete set of representatives of finitely presented right R-modules. Set $M := \prod_{S_i \in \Gamma} S_i^+$, where $S_i^+ \cong Hom(S_i, \mathbb{Q}/\mathbb{Z})$. $\mathcal{P}(M) = \mathcal{F}lat$.

Projective Test Modules

M is called projective test module (shortly, p-test) in case $\mathcal{P}rj = \mathcal{P}(M)$. Whether p-test module exist or not for each ring is unknown.

Let M be a left R-module.

$$\mathcal{I}(M) = \{N \in R-Mod | \operatorname{Ext}^1_R(M,N) = 0\}$$

(1) $\mathcal{I}(M) = R-Mod$ iff M is projective.

(2) Let $\mathcal{I}nj$ be denote the class of all injective modules. $\mathcal{I}nj \subseteq \mathcal{I}(M)$ for each module M.

(3) Let Γ be a complete set of representatives of finitely presented left R-modules. Set $F := \bigoplus_{S_i \in \Gamma} S_i$. $\mathcal{I}(M) = \{N \in R-Mod | N \text{ is fp-injective}\}$.

(4) Let Γ be the set of all proper (essential) right ideals of R. Put $M := \bigoplus_{I \in \Gamma} R/I$. $\mathcal{I}(M) = \mathcal{I}nj$.

Injective Test Modules

M is called injective test module (shortly, i-test) in case $\mathcal{I}nj = \mathcal{I}(M)$.

i-test modules and p-test modules introduced and studied by Jan Trlifaj in several papers. The question ”Is $\mathbb{Z}$ a p-test $\mathbb{Z}$-module?” is exactly the well-known Whitehead problem. Trlifaj defined a ring R to be Ext-ring if each module is either p-test or injective. Trlifaj also considered rings over which each (finitely generated) module is either i-test or projective, and referred to such rings as (n-saturated rings) fully saturated rings. Fully saturated rings and Ext-rings are the same.

Whitehead test modules

1. J. Trlifaj, *von Neumann regular rings and the Whitehead property of modules*, Comment. Math. Univ. Carolin.,(1990).
2. J. Trlifaj, *Whitehead test modules*, Trans. Amer. Math. Soc.,(1996).
3. P. C. Eklof and J. Trlifaj, *How to make Ext vanish*, Bull.

Subprojectivity Domains

A module M is said to be N-subprojective if for every epimorphism $g : B \rightarrow N$ and homomorphism $f : M \rightarrow N$, then there exists a homomorphism $h : M \rightarrow B$ such that $gh = f$, i.e. $\mathrm{Hom}(M, B) \rightarrow \mathrm{Hom}(M, N) \rightarrow 0$.

Subprojectivity Domains

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For a module M, the subprojectivity domain of M, $\underline{\mathfrak{Pr}}^{-1}(M)$, is defined to be the collection of all modules N such that M is N-subprojective, that is
$\underline{\mathfrak{Pr}}^{-1}(M) = \{N \in Mod - R | \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, N) \to 0$ for every epimorphism $g : B \to N\}$.

(1) $\underline{\mathfrak{Pr}}^{-1}(M) = R-Mod$ iff M is projective.

(2) $\mathcal{P}rj \subseteq \underline{\mathfrak{Pr}}^{-1}(M)$ for each module M.

(3) Let Γ be a complete set of representatives of finitely presented left R-modules. Set $F := \prod_{S_i \in \Gamma} S_i$. $\underline{\mathfrak{Pr}}^{-1}(F) = \mathcal{F}lat$.

p-Indigent Modules

The smallest possible subprojectivity domain is the class of projective modules. A module M is called p-indigent in case $\mathcal{P}rj = \underline{\mathfrak{Pr}}^{-1}(M)$. Whether p-indigent module exist or not for each ring is unknown.

1. C. Holston and S. R. López-Permouth and J. Mastromatteo and J. E. Simental-Rodriguez, An alternative perspective on projectivity of modules, *Glasgow Math. J.*, **57(1)** (2015) 83–99.
2. Y. Durğun, Rings whose modules have maximal or minimal subprojectivity domain, *J. Algebra Appl.*, (2015), 14(6).

p-Indigent Modules

1. Subprojectivity domains and p-indigent modules are introduced by C. Holston and S. R. López-Permouth and J. Mastromatteo and J. E. Simental-Rodriguez.

p-Indigent Modules

1. Subprojectivity domains and p-indigent modules are introduced by C. Holston and S. R. López-Permouth and J. Mastromatteo and J. E. Simental-Rodriguez.
2. In his paper, Durğun interested with the structure of rings over which every (finitely generated,singular, simple) module is projective or p-indigent.

Theorem (Durgun,2015)

Let R be a ring which is not von Neumann regular. R is right fully saturated, i.e. every right R-module is projective or i-test iff every right R-module is projective or p-indigent.

By replacing the functor Ext with the functor Tor, we introduced flat version of the Whitehead test modules for injectivity (projectivity) of Trlifaj. In contrast with *Ext*, we could focus on one module. Let M be a right R-module.

$$\mathcal{F}(M) = \{N \in R - Mod | \operatorname{Tor}_1^R(M, N) = 0\}$$

1. $\mathcal{F}(M) = R - Mod$ iff M is flat.
2. $\mathcal{F}lat \subseteq \mathcal{F}(M)$ for each module M.
3. Let Γ be a complete set of representatives of finitely presented right R-modules. Set $F := \prod_{S_i \in \Gamma} S_i$. $\mathcal{F}(M) = \mathcal{F}lat$.

Flat Test Module

M is called flat test module (shortly, f-test) in case $\mathcal{F}lat = \mathcal{F}(M)$.

By replacing the functor $\mathrm{Hom}(M, -)$ with the functor $M \otimes_R -$, we introduced flat version of the subprojectivity domain.
Let M be a right R-module.

$\underline{\mathfrak{Fl}}^{-1}(M) = \{N \in R - Mod | 0 \to M \otimes_R Kerg \to M \otimes_R N$ for every epimorphism $g : B \to N\}$.

Lemma

$\underline{\mathfrak{Fl}}^{-1}(M) = \mathcal{F}(M)$ for each right R-module M.

Proposition

Every i-test module is f-test. In particular, there is f-test module for each ring R.

Proposition

For an arbitrary ring R, the following conditions are equivalent:

(1) R is von Neumann regular.

(2) Every (non-zero) right (left) R-module is f-test.

(3) There exists a right (left) flat f-test R-module.

Theorem

Let R be a nonsemisimple left perfect ring which has at least one finitely generated left maximal ideal (e.g. let R be a left noetherian). The following statements are equivalent:

1. Every finitely generated right R-module is flat or f-test.
2. Every finitely generated left R-module is projective or p-indigent.
3. R is a left Σ-CS ring and every finitely generated singular left R-module is p-indigent.
4. There is a ring direct sum $R \cong S \times T$, where S is semisimple artinian ring and T is an indecomposable ring which is either
   - (i) n-saturated matrix ring over a local QF-ring, or
   - (ii) hereditary artinian serial ring with $J(T)^2 = 0$.

For convenience, we will define the following conditions for a ring R:
(F) : Every right R-module is flat or f-test.

Corollary

Let R be a nonsemisimple ring. Then the following are equivalent:

(1) R is a nonsingular left artinian ring which satisfies (F).
(2) R is a right (or left) hereditary fully saturated ring.
(3) $R \cong S \times T$, where S is semisimple artinian ring and T is an indecomposable hereditary artinian serial ring with $J(T)^2 = 0$.

Lemma

Let R be a nonsemisimple left coherent ring. If every finitely generated non-flat right R-module is f-test, then R is left *IF* ring or left semihereditary ring.

Lemma

Let R be a nonsemisimple left coherent ring. If every finitely generated non-flat right R-module is f-test, then R is left IF ring or left semihereditary ring.

Theorem

Let R be a hereditary noetherian ring. The following are equivalent for a non-flat right R-module M:

(1) M is i-test.

(2) M is f-test.

(3) $\mathrm{Hom}(S, M) \neq 0$ for each singular simple right R-module S.

Corollary

An abelain group is f-test (i-test) if and only if it contains a submodule isomorphic to $\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p ranges over all primes.

We sum up our results:

Theorem

Let R be a noetherian ring such that R is not QF.

(1) R satisfies (F).
(2) Every right R-module is flat or i-test.
(3) R is right n-saturated.
(4) R is right hereditary and every singular right R-module is f-test (i-test).
(5) R is right hereditary and every injective right R-module is flat or f-test (i-test).

Theorem

Let R be a noetherian ring such that R is not left (or right) nonsingular. If R satisfies (F) (or every finitely generated non-flat right (left) R-module is f-test), then $R \cong S \times T$, where S is semisimple artinian ring and T is indecomposable matrix ring over a local QF-ring.

R. Alizade and E. Büyükaşik and N. Er, Rings and modules characterized by opposites of injectivity, *J. Algebra* **409** (2014) 182–198.

P. Aydoğdu and S. R. López-Permouth, An alternative perspective on injectivity of modules, *J. Algebra* **338** (2011) 207–219.

R. R. Colby, Rings which have flat injective modules,*J. Algebra* **35** (1975) 239–252.

Y. Durğun, Rings whose modules have maximal or minimal subprojectivity domain, *J. Algebra Appl.* (2015), 14(6).

- C. Holston and S. R. López-Permouth and J. Mastromatteo and J. E. Simental-Rodriguez, An alternative perspective on projectivity of modules, *Glasgow Math. J.*, **57(1)** (2015) 83–99.
- D.V. Huynh, Structure of some Noetherian SI rings, *J. Algebra* **254(2)** (2002) 362–374.
- J. Trlifaj, Whitehead test modules, *Trans. Amer. Math. Soc.* **348(4)** (1996) 1521–1554.
- P. C. Eklof and J. Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc., (2001).